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www.elsevier.com/locate/physletbThe $f(R)$ gravity function of the Linde quintessenceSergei V. Ketov^{a,b,c,*}, Natsuki Watanabe^a^a Department of Physics, Tokyo Metropolitan University, Minami-ohsawa 1-1, Hachioji-shi, Tokyo 192-0397, Japan^b Kavli Institute for the Physics and Mathematics of the Universe (IPMU), The University of Tokyo, Chiba 277-8568, Japan^c Institute of Physics and Technology, Tomsk Polytechnic University, 30 Lenin Ave., Tomsk 634050, Russian Federation

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ABSTRACT

We calculate the $f(R)$ gravity function in the dual gravity description of the quintessence model with a quadratic (Linde) scalar potential and a positive cosmological constant. We find that in the large curvature regime relevant to chaotic inflation in Early Universe, the dual $f(R)$ gravity is well approximated by the (matter) loop-corrected Starobinsky inflationary model. In the small curvature regime relevant to dark energy in the Present Universe, the $f(R)$ gravity function reduces to the Einstein–Hilbert one with a positive cosmological constant.

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1. Introduction

Theoretical models of cosmological inflation (or primordial dark energy) in Early Universe and those of dynamical dark energy (in the Present Universe) are known to be easily constructed by the use of modified $f(R)$ gravity or quintessence. The standard treatment usually includes the remarkable duality between an $f(R)$ -gravity model and the classically equivalent scalar–tensor gravity (quintessence) model by the Legendre–Weyl transform from the Jordan frame to the Einstein frame, with the standard (quintessence) cosmology in terms of the dual (inflaton) scalar potential (see Section 2 or reviews [1–5] for details).

The most economical, simple and viable inflationary model on the $f(R)$ gravity side is given by the Starobinsky model [6–9], with an action

$$S[g] = \int d^4x \sqrt{-g} \left[-\frac{1}{2}R + \frac{1}{12M^2}R^2 \right], \quad (1.1)$$

in terms of 4D spacetime metric $g_{\mu\nu}(x)$ of the scalar curvature R . We use the natural units with the reduced Planck mass $M_{\text{Pl}} = 1$, unless is otherwise stated. Slow-roll inflation takes place in the high-curvature regime (with $M_{\text{Pl}} \gg H \gg M$ and $|\dot{H}| \ll H^2$), where the second term in Eq. (1.1) dominates. The inflationary model (1.1) has the only mass parameter M that is fixed by the observational Cosmic Microwave Background (CMB) data as $M = (3.0 \times$

$10^{-6})(\frac{50}{N_e})$ where N_e is the e-foldings number. The predictions of the Starobinsky model for the spectral indices $n_s \approx 1 - 2/N_e \approx 0.964$, $r \approx 12/N_e^2 \approx 0.004$ and low non-Gaussianity are in agreement with the WMAP and PLANCK data ($r < 0.13$ and $r < 0.11$, respectively, at 95% CL) [10], but are in apparent *disagreement* with the BICEP2 measurements ($r = 0.2 + 0.07, -0.05$) [11], even after the dust contribution adjustment [12].

The action (1.1) can be dualized by the Legendre–Weyl transform [13,14] to the standard (quintessence) action of the Einstein gravity coupled to a single (canonically normalized) physical scalar (inflaton or scalaron) ϕ having the scalar potential

$$V(\phi) = \frac{3}{4}M^2(1 - e^{-\sqrt{\frac{2}{3}}\phi})^2. \quad (1.2)$$

The exit from the Starobinsky inflation goes to a Minkowski vacuum, though a small positive cosmological constant can always be added in order to shift the Minkowski vacuum to a de-Sitter vacuum that does not affect the inflation.

On the quintessence side, the simplest inflationary model with a *quadratic* scalar potential was proposed by Linde [15]. It predicts $r \approx 8/N_e = 0.16(\frac{50}{N_e})$ in good agreement with the BICEP2 data [11]. By adding a small cosmological constant to the Linde scalar potential one can also take into account the present dark energy (after an exit from the Linde inflation and inflaton decay).

To the best of our knowledge, the $f(R)$ gravity function for the Linde scalar potential is unknown. The main purpose of this letter is to fill this gap in the literature. We calculate that function and find it to be non-trivial, being related to the loop-corrected Starobinsky model (Section 4).

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Our paper is organized as follows. In Section 2 we review the Legendre–Weyl transform and give its inverse form. In Section 3 we find the exact $f(R)$ gravity function for the Linde scalar potential in a parametric form, and analytically study its limits in the cases of high and low spacetime scalar curvature, respectively. Section 4 is our conclusion.

Throughout this paper we use the natural units, $c = \hbar = 1$, and the space–time signature $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The Einstein–Hilbert action with a cosmological constant Λ reads

$$S_{\text{EH}} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (R + 2\Lambda), \quad (1.3)$$

where R is the scalar curvature, $\kappa^2 = \frac{1}{M_{\text{Pl}}^2} = 1.7 \times 10^{-37} \text{ GeV}^{-2}$, and $M_{\text{Pl}} = (8\pi G_{\text{N}})^{-1/2}$ is the (reduced) Planck mass in terms of the Newton constant G_{N} . In our notation, the cosmological constant Λ is positive and the scalar curvature is negative in a *de-Sitter* (dS) spacetime (like the Present Universe), and vice versa in an Anti-de-Sitter (AdS) spacetime.

2. The Legendre–Weyl transform and its inverse

An $f(R)$ gravity action

$$S_f = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) \quad (2.1)$$

subject to the classical and quantum stability conditions (or, equivalently, no ghosts and tachyons, respectively) [1–5]

$$f'(R) < 0 \quad \text{and} \quad f''(R) > 0, \quad (2.2)$$

where the primes denote differentiations, is classically equivalent to

$$S[g_{\mu\nu}, \chi] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [f'(\chi)(R - \chi) + f(\chi)] \quad (2.3)$$

with the real scalar field χ , provided that $f'' \neq 0$ that we always assume. The equivalence can be easily verified because the χ -field equation implies $\chi = R$. The first condition (2.2) also guarantees the existence of the dual (quintessence) description.

The factor f' in front of the R in Eq. (2.3) can be eliminated by a Weyl transformation of metric $g_{\mu\nu}$, so that one can transform the action (2.3) into the action of the scalar field χ minimally coupled to the Einstein gravity and having the scalar potential

$$V = \frac{\chi f'(\chi) - f(\chi)}{2\kappa^2 f'(\chi)^2}. \quad (2.4)$$

The kinetic term of χ becomes canonically normalized after the field redefinition

$$f'(\chi) = -\exp\left(-\sqrt{\frac{2}{3}}\kappa\phi\right) \quad (2.5)$$

in terms of the new scalar field ϕ . As a result, the action $S[g_{\mu\nu}, \chi(\phi)]$ takes the standard quintessence form.

Differentiating the scalar potential V in Eq. (2.4) with respect to ϕ yields

$$\begin{aligned} \frac{dV}{d\phi} &= \frac{dV}{d\chi} \frac{d\chi}{d\phi} \\ &= \frac{1}{2\kappa^2} \left[\frac{\chi f'' + f' - f'}{f'^2} - 2 \frac{\chi f' - f}{f'^3} f'' \right] \frac{d\chi}{d\phi}, \end{aligned} \quad (2.6)$$

where we have

$$\frac{d\chi}{d\phi} = \frac{d\chi}{df'} \frac{df'}{d\phi} = \frac{df'}{d\phi} / \frac{df'}{d\chi} = -\sqrt{\frac{2}{3}} \kappa \frac{f'}{f''}. \quad (2.7)$$

It implies that

$$\frac{dV}{d\phi} = \frac{\chi f' - 2f}{\sqrt{6}\kappa f'^2}. \quad (2.8)$$

Combining Eqs. (2.4) and (2.8) yields R and f in terms of the scalar potential V as follows:

$$R = -\left(-\sqrt{6}\kappa \frac{dV}{d\phi} + 4\kappa^2 V\right) \exp\left(-\sqrt{\frac{2}{3}}\kappa\phi\right), \quad (2.9)$$

$$f = \left(-\sqrt{6}\kappa \frac{dV}{d\phi} + 2\kappa^2 V\right) \exp\left(-2\sqrt{\frac{2}{3}}\kappa\phi\right). \quad (2.10)$$

These two equations define the function $f(R)$ in the parametric form, in terms of a given scalar potential $V(\phi)$.

In the case of the Higgs-like (or, more precisely, the uplifted W -shape) scalar potential for the present dark energy, the corresponding $f(R)$ gravity function was found in Ref. [16].

3. The $f(R)$ gravity dual of the Linde quintessence

We are now in a position to compute the $f(R)$ gravity function for the Linde scalar potential of a canonically normalized inflaton ϕ [15],

$$V_{\text{L}}(\phi) = \frac{m^2}{2} \phi^2 + V_0, \quad (3.1)$$

in order to get its dual (equivalent) gravitational description. The first term in Eq. (3.1) is supposed to dominate during chaotic inflation in Early Universe, whereas the second term (cosmological constant) is supposed to dominate in the Present Universe (dark energy), well after the end of inflation followed by inflaton decay (reheating). Accordingly, the parameters of Eq. (3.1) have to be fixed by current observations as

$$m \approx 6 \times 10^{-6} \quad \text{and} \quad V_0 \approx 10^{-120}. \quad (3.2)$$

It is worth mentioning here that the Linde inflation is consistent with the relatively high (vs. that of the Starobinsky model) tensor-to-scalar ratio r , which is measurable via a detection of the B-mode polarization of the CMB radiation.

In the case of Eq. (3.1), the inverse transform of Section 2 yields

$$R = -3m^2 \left(y^2 - y + \frac{4V_0}{3m^2} \right) e^{-y}, \quad (3.3)$$

$$f = \frac{3}{2} m^2 \left(y^2 - 2y + \frac{4V_0}{m^2} \right) e^{-2y}, \quad (3.4)$$

where we have rescaled the inflaton field as $y = \sqrt{\frac{2}{3}}\phi$. These equations give the exact solution to the $f(R)$ gravity function of the Linde quintessence in the parametric form. Figs. 1 and 2 show the behavior of the function $R(y)$ with a positive V_0 and the vanishing V_0 , respectively, where the factor $3m^2$ is absorbed into the normalization of R and f .

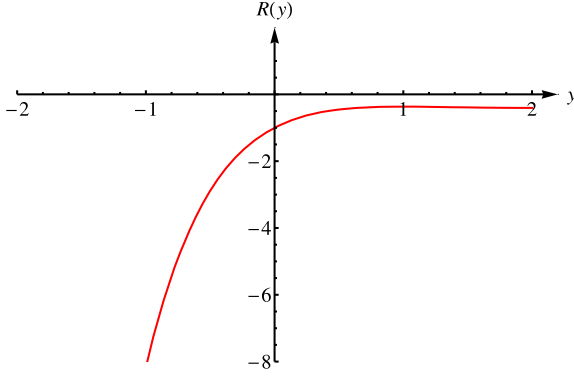
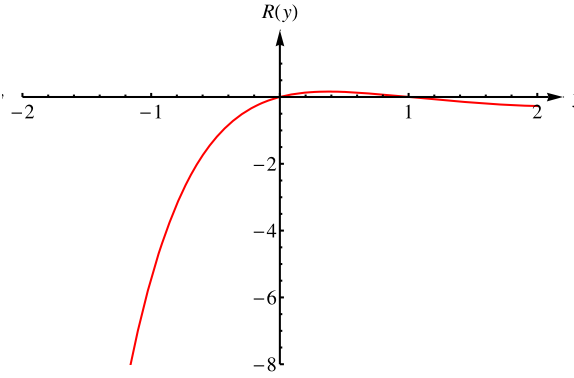
In the large curvature (or large field) approximation relevant to inflation, Eq. (3.3) can be greatly simplified and solved as

$$R \approx -3m^2 e^{-y} \quad \text{and} \quad y(R) \approx -\ln \frac{-R}{3m^2}. \quad (3.5)$$

Substituting the $y(R)$ into Eq. (3.4) yields

$$f^{\text{E}}(R) = \frac{3}{2} m^2 \left(\frac{R}{3m^2} \right)^2 \left[\left(\ln \frac{|R|}{3m^2} \right)^2 + 2 \ln \frac{|R|}{3m^2} + \frac{4V_0}{3m^2} \right], \quad (3.6)$$

where the superscript E refers to the Early Universe. During (slow-roll) inflation, the R^2 term dominates over the R term in Eq. (1.1),

Fig. 1. The case of $V_0 > 0$.Fig. 2. The case of $V_0 = 0$.

so that the $f(R)$ gravity function (3.6) is similar to the Starobinsky function proportional to R^2 , though being corrected by the logarithmic terms (a cosmological constant can be safely ignored during an Early Universe inflation).

It is not difficult to verify the stability conditions (2.2) during inflation, when using our result (3.6). We always assume that the scale of inflation is well below the Planck scale. We find the conditions

$$2m^2 < |R| \ll 1, \quad (3.7)$$

which mean

$$7.2 \times 10^{-11} < \left| \frac{R}{M_{\text{Pl}}^2} \right| \ll 1, \quad (3.8)$$

where we have used Eq. (3.2) and have ignored the V_0 contribution.

The inflationary function $f^E(R)$ can always be represented in the Starobinsky-type form

$$f^E(R) = R^2 A(R) \quad (3.9)$$

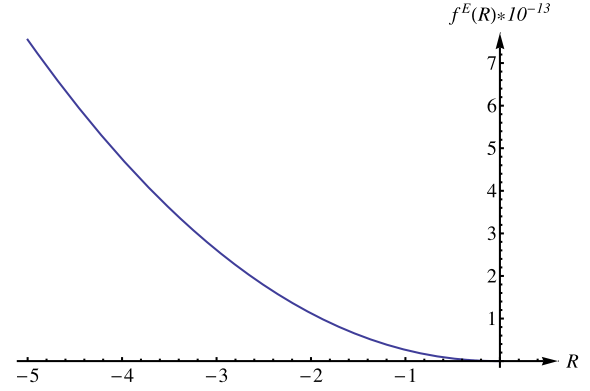
in terms of another (positive) function $A(R)$. A slow-roll inflation of the Starobinsky-type can be achieved by demanding the function $A(R)$ to be “slowly varying” in the sense [5]

$$|A'(R)| \ll \frac{A(R)}{|R|}, \quad (3.10)$$

$$|A''(R)| \ll \frac{A(R)}{R^2}. \quad (3.11)$$

In our case (3.6) we have

$$A(R) = \frac{1}{6m^2} \ln \frac{|R|}{3m^2} \left(\ln \frac{|R|}{3m^2} + 2 \right). \quad (3.12)$$

Fig. 3. The function $f^E(R)$.

We find that the conditions (3.10) and (3.11) imply

$$|R| \gg 3m^2 \quad (3.13)$$

or

$$\left| \frac{R}{M_{\text{Pl}}^2} \right| \gg 1.1 \times 10^{-10}, \quad (3.14)$$

where we have used Eq. (3.2) and have ignored the V_0 contribution again. Hence, the conditions (3.10) and (3.11) are consistent with those of Eq. (3.8).

Similarly, in the *small* curvature (or small field) approximation, we have

$$R = 3m^2 \left(y - \frac{4V_0}{3m^2} \right) + \mathcal{O}(y^2, V_0 y) \quad \text{and}$$

$$y \approx \frac{R + 4V_0}{3m^2}. \quad (3.15)$$

Substituting them into Eq. (3.4) gives rise to the $f(R)$ gravity function

$$f^P(R) = \frac{1}{6m^2} [R^2 - 2(3m^2 - 4V_0)R - 4V_0(3m^2 - 4V_0)] e^{-2\frac{R+4V_0}{3m^2}}, \quad (3.16)$$

where the superscript P refers to the Present Universe. After dropping the terms beyond the first order in R and V_0 , we arrive at the Einstein–Hilbert action with a cosmological constant V_0 , as it should, namely,

$$f^P(R) = -R - 2V_0 + \mathcal{O}(R^2, V_0^2, V_0 R). \quad (3.17)$$

The profiles of the functions $f^E(R)$ and $f^P(R)$ are given in Figs. 3 and 4 with the rescaled argument R .

Checking the stability conditions (2.2) in the low curvature approximation with the $f(R)$ gravity function (3.16) results in the condition

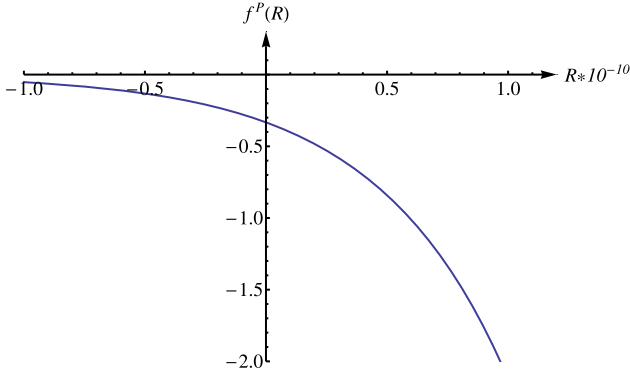
$$R < 1.1m^2, \quad (3.18)$$

where we have used Eq. (3.2) and have ignored the V_0 contribution too. It implies

$$\frac{R}{M_{\text{Pl}}^2} < 4 \times 10^{-11} \quad (3.19)$$

that is obviously satisfied in our Present Universe.

Of course, stability is *not* an issue in the case of the Linde scalar potential (3.1) with $m^2 > 0$ and $V_0 \geq 0$. However, a stability analysis becomes non-trivial on the dual gravity side. The bounds on the scalar curvature found above should be merely considered as the restrictions on the approximation to be used.

Fig. 4. The function $f^P(R)$.

4. Conclusion

Our main new results are given by Eqs. (3.3), (3.4), (3.6) and (3.16). We also verified the stability conditions and gave the profiles of the relevant $f(R)$ gravity functions.

It is remarkable that our result (3.6) for the $f(R)$ gravity function of the Linde inflation in the large curvature regime takes the form of the quantum-corrected Starobinsky inflationary model, with the logarithms representing the *loop* corrections of matter fields in curved space-time, just in the original spirit of the Starobinsky inflation [17].¹ For example, the multi-loop (renormalization-group-improved) *Ansatz* for such quantum corrections was proposed in Ref. [19] in the form

$$f^M(R) = \mu^{-1} R^2 \left[1 + \gamma \ln \left(\frac{R^2}{\mu^2} \right) \right]^{-1} \quad (4.1)$$

with the renormalization group scale μ and the parameter γ . Its expansion in powers of γ leads to all powers of the logarithm in the pre-factor of R^2 , in all the loop-orders.

Eq. (3.6) is also to be compared with the *Ansatz* of the one-loop corrected Starobinsky function (in our notation)

$$f^D(R) = -R + \alpha R^2 + \beta R^2 \log(-R) \quad (4.2)$$

with some coefficients α and β , used in Ref. [20] for the purpose of enhancement of the tensor-to-scalar ratio r of the simplest Starobinsky model (1.1) towards its matching with the BICEP2 data [11]. As was demonstrated in Ref. [20], despite the enhancement in r , the function (4.2) is disfavored by the BICEP2 measurements for any values of the coefficients α and β . However, as is clear from a simple comparison of Eq. (4.2) with our Eq. (3.6), in order to reach a viable enhancement, one should also take into account the quantum corrections given by the logarithm *squared*, of the type $R^2 \log^2(-R)$, which arise from the higher loops (or the two loops, at least), as above.

Thus, it follows from our results that the two-loop corrected Starobinsky model, contrary to the one-loop corrected one, can

provide the enhancement of the tensor-to-scalar ratio r to the BICEP2 value.

Finally, it is worth mentioning that the considerations of this paper allow an extension to the $N = 1$ supergravity, by using curved superspace – see e.g., Refs. [21,22].

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¹ See also Ref. [18] for some explicit examples of such quantum corrections.